SADLER UNIT 4 MATHEMATICS METHODS

WORKED SOLUTIONS

Chapter 4 The normal distribution

Exercise 4A

Question 1

a $\frac{65-60}{5}=1$

b $\frac{72-55}{10}=1.7$

c $\frac{50-58}{4}=-2$

d $\frac{60-58}{4}=0.5$

e $\frac{58-64}{8}=-0.75$

Question 2

$\text{Test A}: \frac{30-20}{4}=2.5$

$\text{Test B}: \frac{50-60}{10}=-1$

$\text{Test C}: \frac{7-6}{0.8}=1.25$

$\text{Test D}: \frac{26-25}{5}=0.2$

Question 3

Mathematics

$$\frac{56-60}{10.4} = -0.385$$

Chemistry

$$\frac{74-72}{7.2} = 0.278$$

Electronics

$$\frac{39-48}{14.6} = -0.616$$

Computing

$$\frac{72-63}{7.4} = 1.216$$

$\therefore$ Computing, Chemistry, Mathematics, Electronics

Question 4

Subject	x	$\bar{x}$	σ	z
Mathematics	76	63	14	$\frac{76-63}{14} = 0.93$
English	75	64	10	$\frac{75-64}{10} = 1.1$
Science	78	72	8	$\frac{78-72}{8} = 0.75$
Social Studies	104	106	22	$\frac{104-106}{22} = -0.09$

$\therefore$ English, Mathematics, Science, Social Studies

Question 5

$x = 65.2,\ \sigma = 8.8$

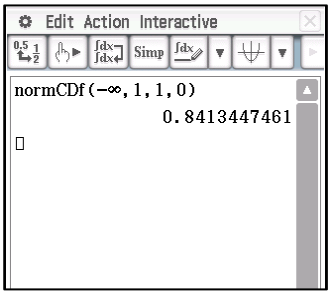
$\text{Jill}: \frac{74}{120}$ $\quad$ $\text{Jack}: \frac{63}{120}$

a 1

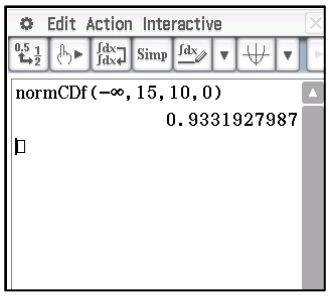
b 0

c -0.25

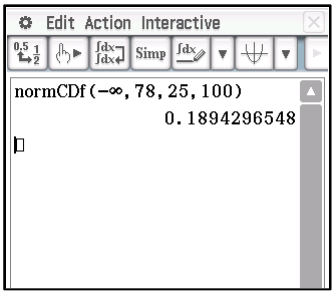
Question 1



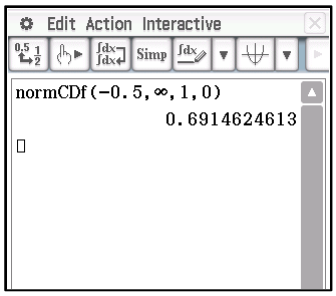
Question 2



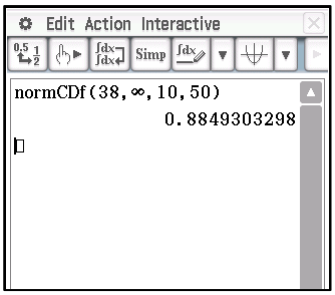
Question 3



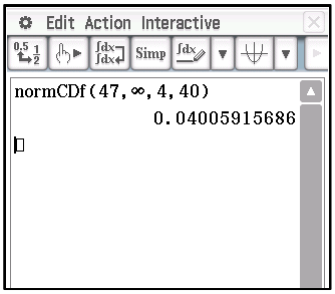
Question 4



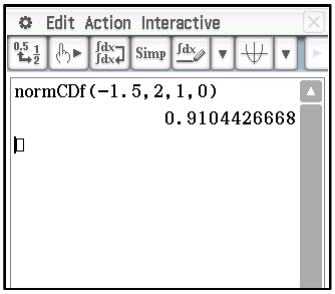
Question 5



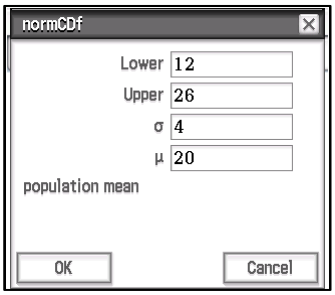
Question 6

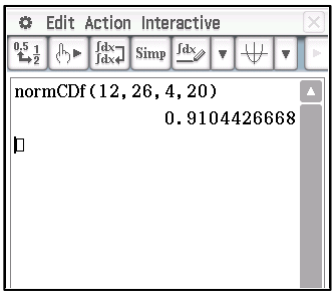


Question 7

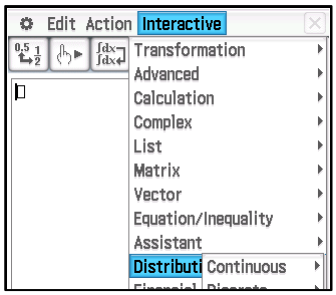


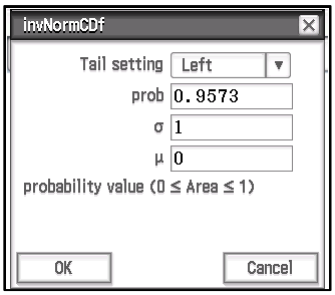
Question 8

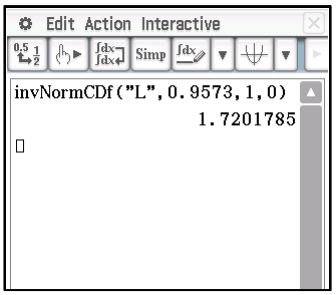




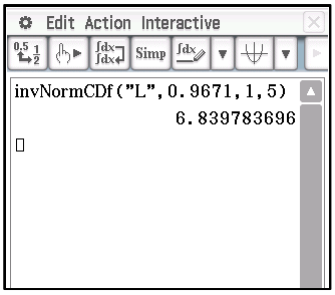
Question 9



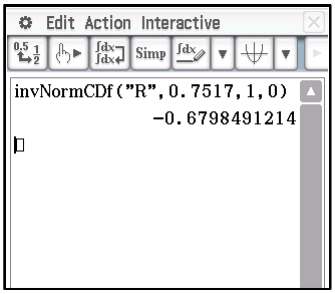




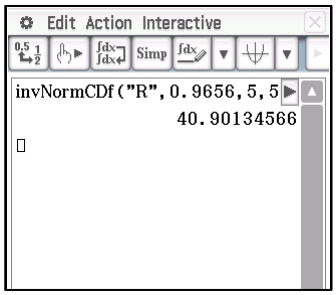
Question 10



Question 11



Question 12



Question 13

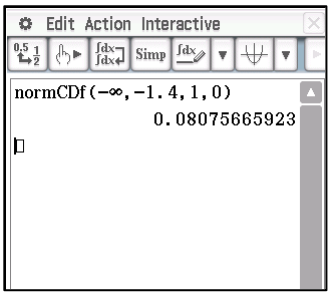
$\mathrm{P}(X < -1.4) = 0.0808$

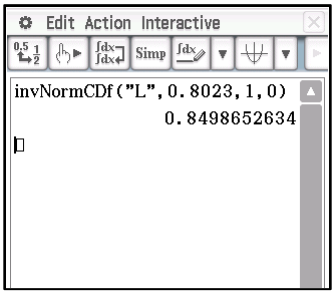
$\mathrm{P}(-1.4 < X < k) = \mathrm{P}(X < k) - \mathrm{P}(X < -1.4)$

$0.7215 = \mathrm{P}(X < k) - 0.0808$

$\mathrm{P}(X < k) = 0.8023$

$k = 0.85 \ \ (2\text{dp})$





Question 14

$\mathrm{P}(X < 87.2) = 0.0808$

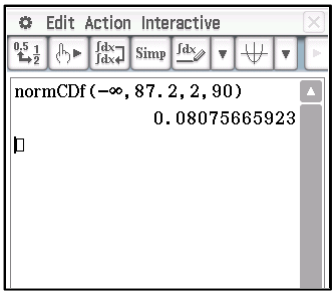
$\mathrm{P}(87.2 < X < k) = \mathrm{P}(X < k) - \mathrm{P}(X < 87.2)$

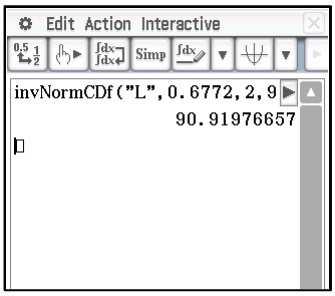
$\mathrm{P}(87.2 < X < k) = \mathrm{P}(X < k) - \mathrm{P}(X < 87.2)$

$0.5964 = \mathrm{P}(X < k) - 0.0808$

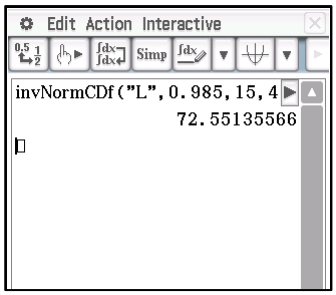
$0.6772 = \mathrm{P}(X < k)$

$k = 90.92 \ (2\text{dp})$

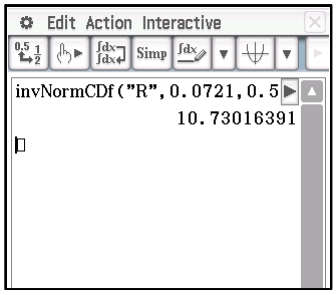




Question 15



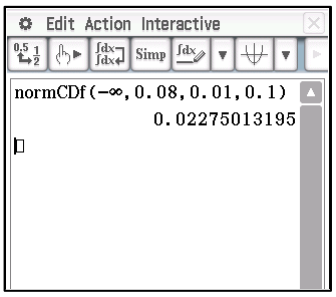
Question 16



Question 17

$$\mathrm{P}(0.08 < X < k) = \mathrm{P}(X < k) - \mathrm{P}(X < 0.08)$$

$$\mathrm{P}(X < 0.08) = 0.0228$$

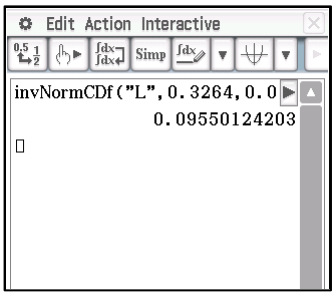


$$\mathrm{P}(0.08 < X < k) = \mathrm{P}(X < k) - \mathrm{P}(X < 0.08)$$

$$0.3036 = \mathrm{P}(X < k) - 0.0228$$

$$\mathrm{P}(X < k) = 0.3264$$

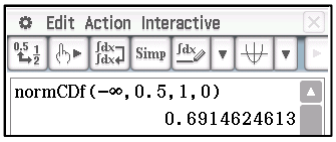
$$k = 0.0955 \text{ (4dp)}$$



Exercise 4C

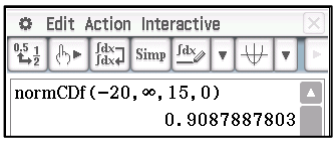
Question 1

$\mathrm{P}(X < 0.5) = 0.6915$



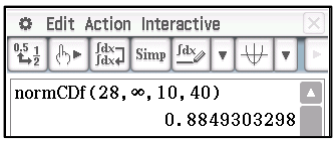
Question 2

$\mathrm{P}(X > -20) = 0.9088$



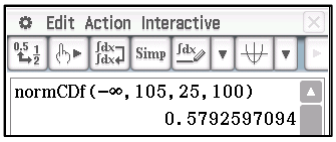
Question 3

$\mathrm{P}(X > 28) = 0.8849$



Question 4

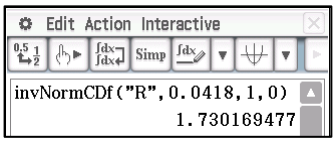
$\mathrm{P}(X < 105) = 0.5793$



Question 5

$\mathrm{P}(X > k) = 0.0418$

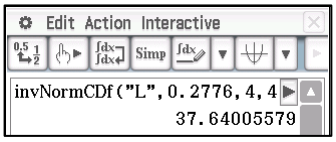
$k = 1.73$



Question 6

$\mathrm{P}(X < k) = 0.2776$

$k = 37.64$



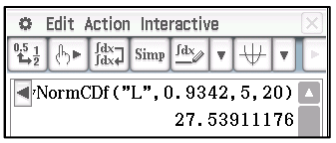
Question 7

$\mathrm{P}(X < 20 + k) = 0.4342 + 00.5$

$= 0.9342$

$20 + k = 27.54$

$k = 7.54$



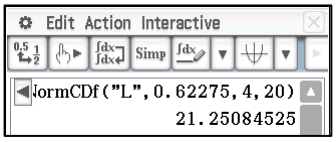
Question 8

$\mathrm{P}(12 < X < k) = 0.6$

$\mathrm{P}(X < 12) = 0.02275$

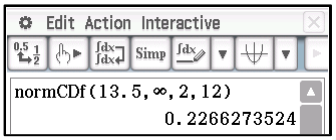
$\mathrm{P}(X < k) = 0.62275$

$\therefore k = 21.25$



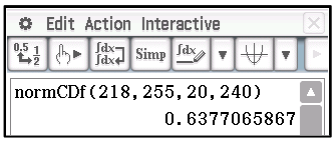
Question 9

$\mathrm{P}(X \geq 13.5) = 0.2266$



Question 10

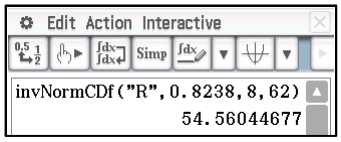
$\mathrm{P}(218 < X < 255) = 0.6377$



Question 11

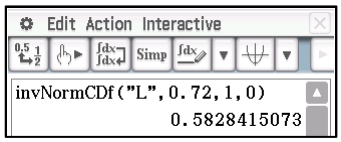
$\mathrm{P}(X > k) = 0.8238$

$k = 54.56$

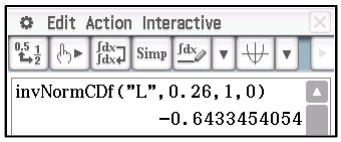


Question 12

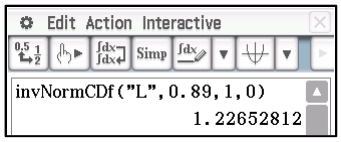
a $\mathrm{P}(X < 0.72) = 0.5828$



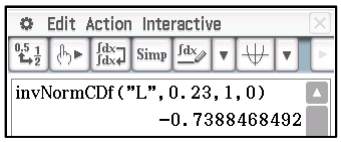
b $\mathrm{P}(X < 0.26) = -0.6433$



c $\mathrm{P}(X < 0.89) = 1.2265$

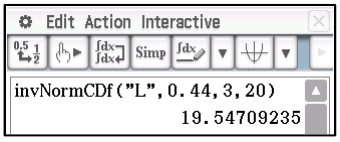


d $\mathrm{P}(X < 0.23) = -0.7388$

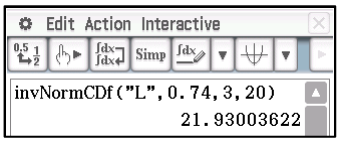


Question 13

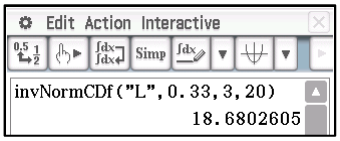
a $P(X < 0.44) = 19.5$



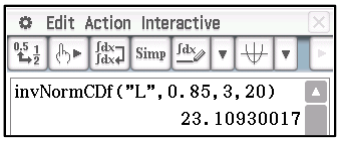
b $P(X < 0.74) = 21.9$



c $P(X < 0.33) = 18.7$



d $P(X < 0.85) = 23.1$



Question 14

a Each score is 1 standard deviation away from the mean: 68% = 0.68

b Each score is 2 standard deviations from the mean: 95% = 0.95

c Each score is 3 standard deviations from the mean: 99.7% = 0.997

d Distribution has a mean of 20 and $\sigma = 6$

$$\frac{8-20}{6} = -2 \qquad \frac{32-20}{6} = 2$$

The scores are 2 standard deviations either side of the mean: 0.95

e Distribution has a mean of 10 and $\sigma = 2$

$$\frac{4-10}{2} = -3 \qquad \frac{16-10}{2} = 3$$

The scores are 3 standard deviations either side of the mean: 0.997

f Distribution has a mean of 0 and $\sigma = 1$

$0.5 \times 0.68 = 0.34$

g Distribution has a mean of 0 and $\sigma = 1$

$$\begin{aligned} P(X < 1) &= P(X < 0) + P(0 < X < 1) \\ &= 0.5 + 0.34 \\ &= 0.84 \end{aligned}$$

h Distribution has a mean of 0 and $\sigma = 1$

$$\begin{aligned} P(X > 1) &= 1 - P(X < 1) \\ &= 1 - 0.84 \\ &= 0.16 \end{aligned}$$

i Distribution has a mean of 0 and $\sigma = 5$

5 is one standard deviation above the mean

$$P(X < 5) = 0.84$$

j Distribution has a mean of 60 and $\sigma = 10$

70 is one standard deviation above the mean

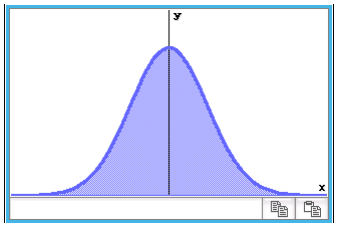
$$P(X > 70) = 0.16$$

Question 15

Let X represent the number of days duration for a naturally delivered human baby. $X \sim (280, 10^2)$

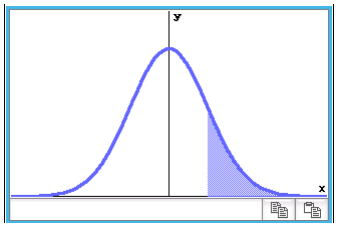
a $\dfrac{250 - 280}{10} = -3$ $\qquad$ $\dfrac{310 - 280}{10} = 3$

$$P(250 < X < 310) = 99.7\%$$



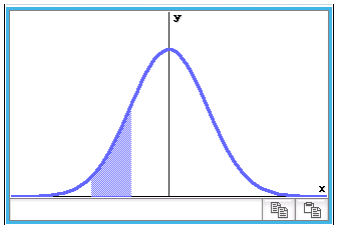
b $\dfrac{290 - 280}{10} = 1$

$$\begin{aligned} P(X > 290) &= 1 - P(X < 290) \\ &= 1 - 84\% \\ &= 16\% \end{aligned}$$



c $\dfrac{260 - 280}{10} = -2$ $\qquad$ $\dfrac{270 - 280}{10} = -1$

$$\begin{aligned} P(260 < X < 270) &= P(260 < X < 280) - P(270 < X < 0) \\ &= \frac{95\%}{2} - \frac{68\%}{2} \\ &= 13.5\% \end{aligned}$$



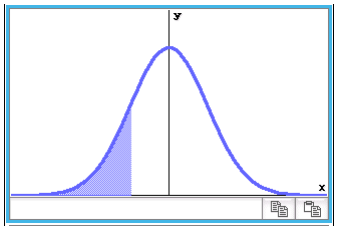
Question 16

Let X represent the weights of the components.

$X \sim (500, 5^2)$

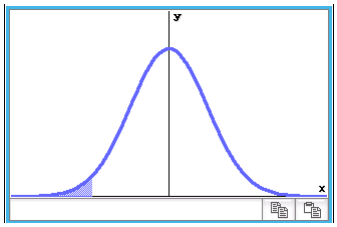
a $\dfrac{495-500}{5} = -1$

$$\begin{aligned} P(X<-1) &= P(X<0) - P(0<X<-1) = \\ &= 50\% - 34\% \\ &= 16\% \end{aligned}$$



b $\dfrac{490-500}{5} = -2$

$$\begin{aligned} P(X<-1) &= P(X<0) - P(0<X<2) = \\ &= 50\% - \frac{95}{2}\% \\ &= 2.5\% \end{aligned}$$

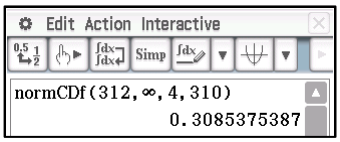


Question 17

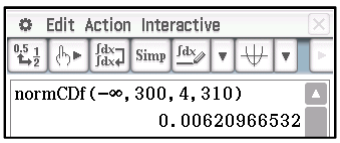
Let X represent the weight of cereal contain in the box.

$X \sim (310, 4^2)$

a $P(X > 312) = 0.3085$



b $P(X < 300) = 0.0062$

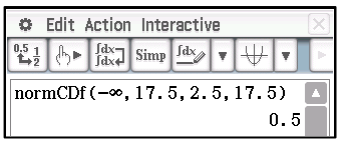


Question 18

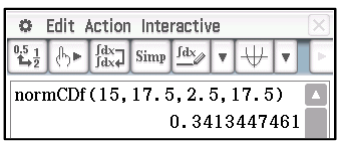
Let X represent the lenths of the adult male lizards.

$X \sim (17.5, 2.5^2)$

a $P(X < 17.5) = 0.5$



b $P(15 < X < 17.5) = 0.34$

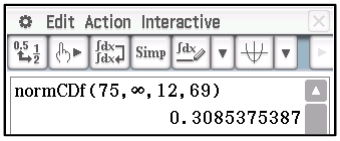


Question 19

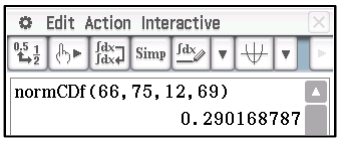
Let X represent the scaled scores in a national mathematics test.

$X \sim (69, 12^2)$

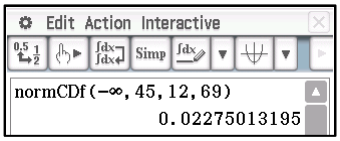
a $\mathrm{P}(X > 75) = 0.3085$



b $\mathrm{P}(66 < X < 75) = 0.2902$



c $\mathrm{P}(X < 45) = 0.0228$



Question 20

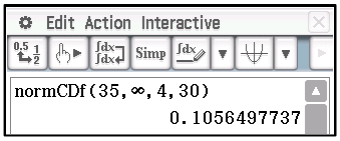
Let X represent the heights of fully grown plants.

$X \sim (30, 4^2)$

a $\mathrm{P}(X > 35) = 0.1056$

$0.1056 \times 100 = 10.56$

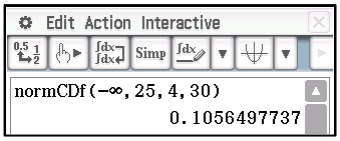
Approximately 11 plants



b $\mathrm{P}(X < 25) = 0.1056$

$0.1056 \times 100 = 10.56$

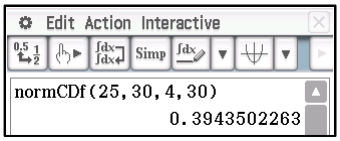
Approximately 11 plants



c $\mathrm{P}(25 < X < 30) = 0.3944$

$0.3944 \times 100 = 39.44$

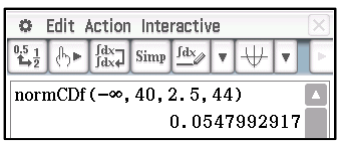
Approximately 39 plants



Question 21

Let X represent the weight of the vitamin contained in 110 mL containers.

$X \sim (44, 2.5^2)$



$110\% \times RDI = 44$

$RDI = 40$ mg

$\mathrm{P}(X < 40) = 0.0548$

Question 22

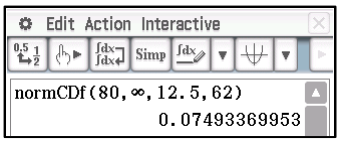
Let X represent the scores obtained on a particular leaving exam.

$X \sim (62, 12.5^2)$

a $\mathrm{P}(X > 80) = 0.0749$

$0.0749 \times 5542 = 415.28$

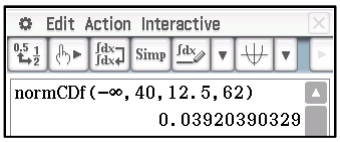
Approximately 415 students



b $\mathrm{P}(X < 40) = 0.0392$

$0.0392 \times 5542 = 217.27$

Approximately 217 students



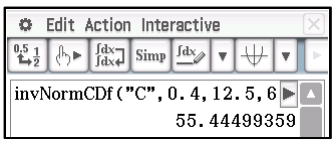
c $P(62 - k < X < 62 + k) = 0.4$

$62 - k = 55.4$

$k = 6.6$

$62 + 6.6 = 68.6$

Stated to the nearest half mark, the middle 40% of scores fall between 55.5 and 68.5



Question 23

Let X represent the heights of adults.

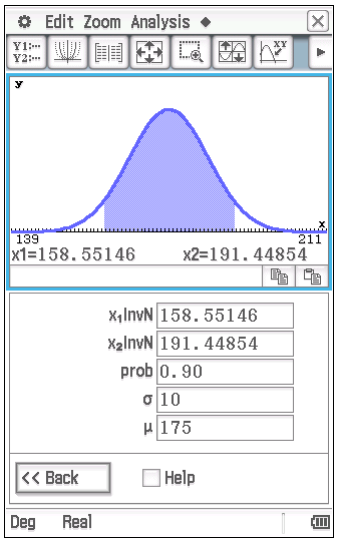
$X \sim (175, 10^2)$

$\mathrm{P}(a < X < b) = 0.90$

By classpad,

$a = 158.5, b = 191.5$

The shortest driver is 158.5 cm and the tallest 191.5 cm.

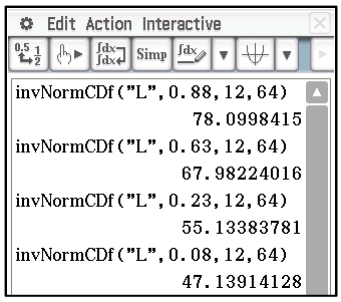


Question 24

Let X represent the marks acheived in the exam.

$X \sim (64, 12^2)$

The top 12% corresponds to the 88th percentile (12% scores higher



$P(X < k) = 0.88$

$k = 78.0998$

The A/B cut off is 78%.

The next 25% percent corresponds to the 63rd percentile

$P(X < k) = 0.63$

$k = 67.9822$

The B/C cut off is 68%.

$P(X < k) = 0.23$

$k = 55.1338$

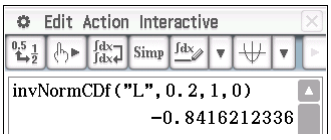
The C/D cut off is 55%.

$P(X < k) = 0.08$

$k = 47.1391$

The D/F cut off is 47%.

Question 25



a $P(X < k) = 0.2$

$\therefore$ 0.84 standard deviations below mean

b $\dfrac{40 - \mu}{5} = -0.84$

$40 - \mu = -4.2$

$\mu = 44.2$

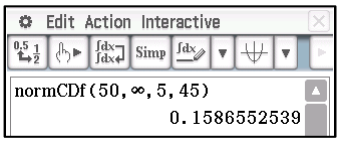
Question 26

Let X represent time in minutes it takes for Monica to arrive at work.

$X \sim (45, 5^2)$

a Monica will need to take more than 50 minutes to be classified as late

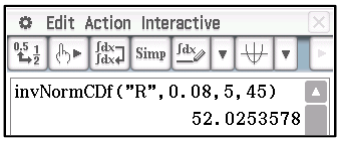
$\mathrm{P}(X > 50) = 0.1587$



b $\mathrm{P}(X < k) = 0.08$

$k = 52.0254$

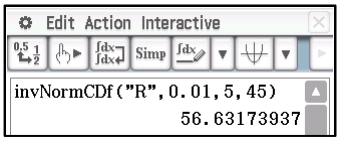
$8:30 - 52.0254 = 7:38$ a.m.



c $\mathrm{P}(X < k) = 0.01$

$k = 56.6317$

$8:30 - 56.63 = 7:33$ a.m.



Question 27

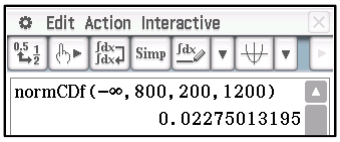
Let X represent the annual rainfall in an area in south west WA.

$X \sim (1200, 200^2)$

a $\text{P}(X < 800) = 0.0228$

$0.0228 \times 100 = 2.28$

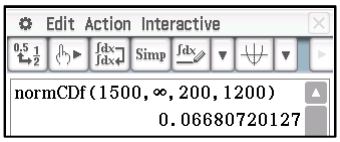
Approximately 2 years



b $\text{P}(X > 1500) = 0.0668$

$0.0668 \times 100 = 6.68$

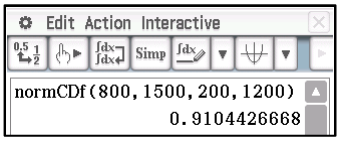
Approximately 7 years



c $\text{P}(800 < X < 1500) = 0.9104$

$0.9104 \times 100 = 91.04$

Approximately 91 years

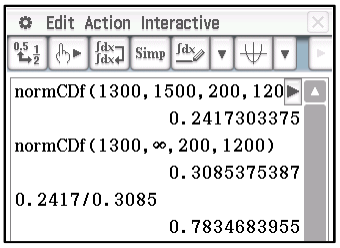


d $\text{P}(X < 1500 \mid X > 1300)$

$$= \frac{\text{P}(1300 < X < 1500)}{\text{P}(X > 1300)}$$

$$= \frac{0.2417}{0.3085}$$

$$= 0.783$$



Question 28

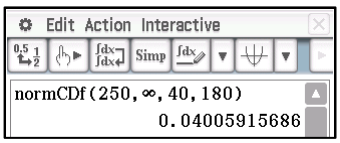
Let X represent the weight of apples from the orchard.

$X \sim (180, 40^2)$

a $\text{P}(X > 250) = 0.0401$

$0.0401 \times 1000 = 40.01$

Approximately 40 apples

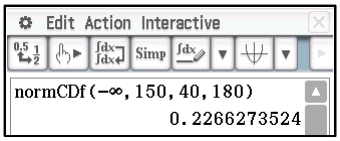


b $\text{P}(X < 150 \mid X < 250)$

$$= \frac{\text{P}(X < 150)}{\text{P}(X < 250)}$$

$$= \frac{0.2266}{1 - 0.0401}$$

$$= 0.2361$$



Question 29

a $\mu = 1018,\ \sigma = 10$

$\text{P}(X < 1000) = 0.0359$

b $\text{P}(X > 1025) = 0.2420$

c $\text{P(success)} = 0.0359$

$\text{P(at least 1)} = 1 - \text{P(none)}$

$\binom{10}{0}(0.0359)^0(0.9641)^{10}$

$= 0.6938$

$1 - 0.6938 = 0.3062$

Question 30

a $\mu = 100,\ \sigma = 15$

$\text{P}(X > 135 \mid X > 125)$

$= \dfrac{\text{P}(X > 135)}{\text{P}(X > 125)}$

$= \dfrac{0.0098}{0.0478}$

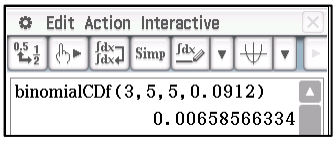
$= 0.2054$

b P(3, 4 or 5 with $X > 120$)

$\text{P}(X > 120) = 0.0912$

Binomial CDf (3, 5, 5, 0.0912)

$= 0.0066$



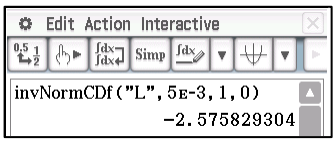
Question 31

a $\mu = x,\ \sigma = 1.8$

Using a standard normal distribtion,

$\text{P}(x < k) = 0.005$

$k = -2.5758$



The labelled weight 500g is -2.5758 standard deviations under the mean

$500 + 2.5758\sigma = x$

$500 + 2.5758(1.8) = 504.636$

$\therefore$ 505 grams

b $250 + 2.5758 \times 1.8 = 254.63$

$\therefore$ 255 grams

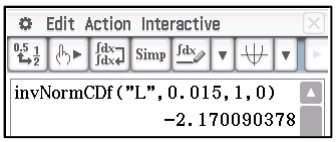
Question 32

a $\mu = 500\,\text{g},\ \sigma = 5\,\text{g}$

$\text{P}(X < 490) = 0.0227$

$\therefore \sim 2.3\%$

b $\mu = 500\ \text{g}$



$\text{P}(X < 490) = 0.015$

A score of -2.17 standard deviations below the mean = 490.

$$\frac{490 - 500}{\sigma} = -2.17$$

$$\sigma = \frac{490 - 500}{-2.17}$$

$$= 4.6$$

Exercise 4D

Question 1

See full answer in text.

Question 2

See full answer in text.

Question 3

See full answer in text.

Question 4

See full answer in text.

Question 5

See full answer in text.

Question 6

See full answer in text.

Question 7

See full answer in text.

Question 8

See full answer in text.

Question 1

$$\frac{d}{dx}\ln(10x) = \frac{1}{x}$$

Question 2

$$\frac{d}{dx}10\ln x = \frac{10}{x}$$

Question 3

$$\frac{d}{dx}\left(\frac{x}{\ln x}\right)$$
$$= \frac{\ln x \times 1 - x \times \frac{1}{x}}{(\ln x)^2}$$
$$= \frac{\ln x - 1}{(\ln x)^2}$$

Question 4

$$\frac{d}{dx}\ln\left[(x^2+1)^3\right]$$
$$= \frac{d}{dx}\left(3\ln(x^2+1)\right)$$
$$= 3 \times \frac{2x}{(x^2+1)}$$
$$= \frac{6x}{x^2+1}$$

Question 5

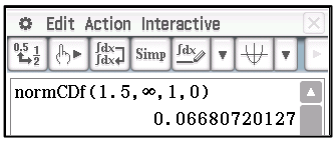
$$\frac{d}{dx}\left[\ln\frac{(x-1)^3}{x+1}\right]$$
$$=\frac{d}{dx}\left(\ln(x-1)^3-\ln(x+1)\right)$$
$$=\frac{d}{dx}\left(3\ln(x-1)-\ln(x+1)\right)$$
$$=\frac{3}{x-1}-\frac{1}{x+1}$$
$$=\frac{3(x+1)-(x-1)}{(x-1)(x+1)}$$
$$=\frac{3x+3-x+1}{(x-1)(x+1)}$$
$$=\frac{2x+4}{(x-1)(x+1)}$$
$$=\frac{2(x+2)}{(x-1)(x+1)}$$

Question 6

$$\frac{d}{dx}\log_5 x$$
$$=\frac{d}{dx}\frac{\ln x}{\ln 5}$$
$$=\frac{1}{\ln 5}\times\frac{1}{x}$$
$$=\frac{1}{x\ln 5}$$

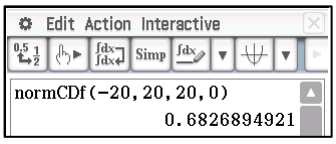
Question 7

$$\mathrm{P}(X>1.5)=0.0668$$



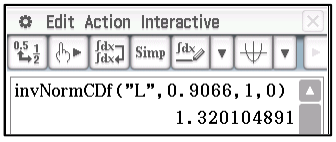
Question 8

$$\mathrm{P}(-20<X<20)=0.6827$$

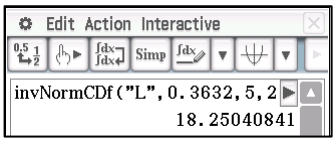


Question 9

$$k=1.32$$



Question 10



$k = 18.25$

Question 11

$$\frac{dy}{dx} = 3 \times \frac{1}{x}$$

When $x = e$

$$\frac{dy}{dx} = \frac{3}{e}$$

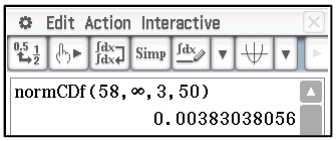
Question 12

$$\frac{dy}{dx} = x \times \frac{1}{x} + \ln x \times 1$$
$$= \ln x + 1$$

When $x = e$,

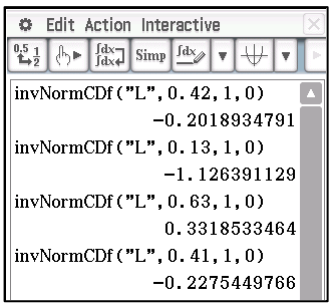
$$\frac{dy}{dx} = \ln e + 1$$
$$= 2$$

Question 13



$P(X \geq 58) = 0.0038$

Question 14



a –0.202

b –1.126

c 0.332

d –0.228

Question 15

Let R represent rainfall

$R \sim N(11.2,\ 3.1^2)$

a $\text{P}(R < 6) = 0.0467$

$0.0467 \times 365 = 17.0565$

$\therefore \approx 17$ days

b $\text{P}(R > 10) = 0.6507$

$0.6507 \times 365 = 237.4900$

$\therefore \approx 237$ days

c $\text{P}(10 < R < 15) = 0.5405$

$0.5405 \times 365 = 197.2905$

$\therefore \approx 197$ days

Question 16

$$\int_0^1 (ax^2 + k)\ dx$$

$$= \left[\frac{ax^3}{3} + kx\right]_0^1$$

$$= \frac{a}{3} + k$$

$$\frac{1}{5} = \frac{a}{3} + k \qquad \rightarrow \text{ Equation 1}$$

$$\int_0^2 (ax^2 + k)\ dx$$

$$= \left[\frac{ax^3}{3} + kx\right]_0^2$$

$$= \frac{8a}{3} + 2k$$

$$\frac{8a}{k} + 2k = 1 \qquad \rightarrow \text{ Equation 2}$$

Solving simultaneously

$a = 0.3,\ k = 0.1$

$$\text{E}(X) = \int_0^2 xf(x)\ dx$$

$$= \int_0^2 (0.3x^2 + 0.1)x\ dx$$

$$= 1.4$$

$$\text{Var}(X) = \int_0^2 f(x)(x - \mu)^2 dx$$

$$= \int_0^2 (0.3x^2 + 0.1)(x - 1.4)^2 dx$$

$$= \frac{17}{75}$$

Question 17

$\int_0^{\pi} k \sin x \ dx = 1$

a

$$\begin{aligned}
\left[-k \cos x\right]_0^{\pi} &= 1 \\
-k \cos \pi - (-k \cos 0) &= 1 \\
-k(-1) - (-k \times 1) &= 1 \\
k + k &= 1 \\
2k &= 1 \\
k &= \frac{1}{2}
\end{aligned}$$

b

$$\begin{aligned}
&\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1}{2} \sin x \ dx \\
&= \left[-\frac{1}{2} \cos x \right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \\
&= \frac{1}{2} \times \frac{\sqrt{2}}{2} - \left(-\frac{1}{2} \times \frac{\sqrt{2}}{2} \right) \\
&= \frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{4} \\
&= \frac{2\sqrt{2}}{4} \\
&= \frac{\sqrt{2}}{2}
\end{aligned}$$

Question 18

a $P(x) = R(x) - C(x)$

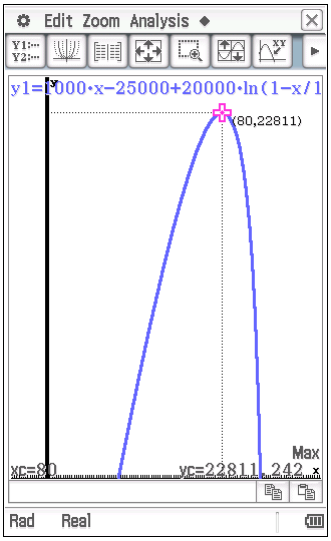
$$= 1000x - (25\,000 - 20\,000\ln\left(1 - \frac{x}{100}\right)),\ x < 100$$

$$= 1000x - 25\,000 + 20\,000\ln\left(1 - \frac{x}{100}\right),\ x < 100$$

b $$P(x) = 1000x - 25\,000 + 20\,000\ln\left(\frac{100 - x}{100}\right)$$

$$= 1000x - 25\,000 + 20\,000\ln\left(\frac{1}{100}(100 - x)\right)$$

$$= 1000x - 25\,000 + 20\,000\left(\ln\frac{1}{100} + \ln(100 - x)\right)$$



$$P'(x) = 1000 + 20\,000\left(\frac{-1}{100 - x}\right),\ x < 100$$

$$0 = 1000 + 20\,000\left(\frac{1}{x - 100}\right)$$

$$-1000 = \frac{20\,000}{x - 100}$$

$$x - 100 = -20$$

$$x = 80$$

$$P(80) = 1000(80) - 25\,000 + 20\,000\ln\left(\frac{20}{100}\right)$$

$$= 22.811.24$$

$$\therefore \approx \$22\,800$$

Question 19

a $\mathrm{P}(X \leq 2) = \dfrac{2^2 + 3(2) - 4}{36}$

$= \dfrac{1}{6}$

b $\mathrm{P}(X \geq 2) = 1 - \dfrac{1}{6}$

$= \dfrac{5}{6}$

c $\mathrm{P}(X \leq 3) = \dfrac{3^2 + 3(3) - 4}{36}$

$= \dfrac{7}{18}$

$\mathrm{P}(3 \leq X \leq 5) = \dfrac{11}{18}$

d 0

Question 20

Let K represent the number of kilometres a new tyre lasts.

$K \sim N(60\ 000, 8000^2)$

a $\mathrm{P}(K < 45\ 000) = 0.0304$

b P(at least one lasts less than 45 000)

$= 1 - \mathrm{P}$(all last more than 45 000)

$= 1 - (1 - 0.0304)^2$

$= 0.1162$